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ON THE EXPRESSION 0^0.

BY PROF. W. W. JOHNSON, ANNAPOLIS, MARYLAND.

It is now a well-established doctrine that 0^0 is one of the indeterminate forms, since, when a function takes this form, its logarithm takes the form $0.\infty$: yet we almost invariably find that in such cases the value of the function is unity; and it is not surprising that, not many years ago, it was generally held that $0^0=1$ was a universally true equation. An interesting controversy on this point took place in Crelle's Journal about the year 1830.

Libri, in an Article on discontinuous functions (Crelle, Vol. X, p. 303), employs the expression 0^0 as equivalent to unity. In support of this he asserts that x log 0 (the log of 0^x) is always 0 when $x = 0$, although the second factor is infinite. He admits that some uncertainty exists, since, admitting that $x \log x = 0$ when $x = 0$, "we do not know that the log 0 arises from log x"; and he proceeds to remove this doubt by a direct proof that $0^0 = 1$. For this purpose he expands $(1 - u)^x$ by the Binomial Theorem, and remaking that for entire values of x the development arrests itself after a finite number of terms by virtue of the occurrence of the factor $x - x$ in succeeding coefficients, he concludes that $(1 - u)^0 = 1$, *independently of the value of* u; and therefore when $u = 1$, we have $0^0 = 1$.

Libri's object in establishing $0^0 = 1$ is, by the way, a very curious one; he wishes to construct a function of x which shall admit of but two values, one corresponding to all positive, and the other to all negative values of x. Taking $0^0 = 1$, he concludes that 0^x admits of but three values 0, 1 and ∞, according as x is positive, 0 or negative: hence x^{0^x} can take only the values $0^0, 0, 0^\infty$, or 1, 0, 0; that is its value is 1 for all positive values of x, and 0 for $x = 0$ and all negative values of x. Libri had indeed previously published this result in Crelle, Vol. VI, p. 67.

In Vol. XI the author of Article 25, "Sur la valeur de 0^0" (p. 272), takes exception to Libri's assumption that $0^0 = 1$, although he says that this is nearly the unanimous opinion of Mathematicians, Cauchy alone, among authors whom he has consulted, dissenting. He quotes from Euler the argument "$\frac{a}{a} = a^0 = 1$ whatever the value of a, hence certainly when $a = 0$;" but the author says why not equally well say $a \div a = 1$, hence, making $a = 0$, $0 \div 0 = 1$ universally. [The argument is that, while we admit that, by the law of continuity, a^0 is unity when $a = 0$, or as we may express it $a^0]_0 = 0$, just as $\left.\frac{a}{a}\right]_0 = 1$; we can no more infer from the former that $0^0 = 1$ universally, than from the latter that $0 \div 0 = 1$ universally. The same remark applies to Libri's deduction from $(1 - u)^0 = 1$; reference to the Binomial Theorem adding nothing whatever to the strength of his position.] To prove $0^0 = 1$, he continues, it is necessary to show that, if $F(x)$ and $f(y)$ vanish for $x = a$ and $y = b$, $[F(x)]^{f(y)}$ becomes unity when $x = a$ and $y = b$. In his opinion 0^0 is nothing more than $0^1 . 0^{-1}$ or $0 . \infty$ which may have any value.

In the next volume of Crelle published in 1834, Möbius gives (XII. p. 134) a proof, communicated to him in 1814 by Pfaff, that $0^0 = 1$, which proof he says in point of validity leaves nothing to be desired, and which he now publishes on account of the doubts, as to the universal truth of this equation, raised by the author of No. 25, Vol. XI. He begins by asserting that $0^0 = 1$ "means nothing but that x^x becomes unity when $x = 0$." Then follows Pfaff's proof of this proposition; to which Möbius adds that, this being established, it is easy to show, as demanded by the author of No. 25, that, if $F(x)$ and $f(x)$ vanish simultaneously, $[F(x)]^{f(x)}$ will take the value unity: in other words, to show that, X and Y being functions of a third variable z which vanish when $z = c$, or more simply when $z = 0$, X^Y becomes unity when $z = 0$.

To prove this he proceeds as follows:—Since for $z = 0$ X and Y vanish, they may be written in the forms,

$$X = Pz^m, \qquad Y = Qz^n,$$

where m and n are positive and independent of z, while P and Q are two functions of z which do not vanish with z, but take certain finite values, say p and q, when $z = 0$. Then

$$X^Y = \left(P^Q\right)^{z^n} \left(z^m\right)^{Qz^n} :$$

when $z = 0$ the first factor becomes $\left(p^q\right)^0 = 1$; and the second, by writing V for z^n, becomes $V^{\frac{m}{n}QV} = \left(V^V\right)^{\frac{m}{n}q}$, which when $z = 0$, or $V = 0$, becomes (according to Pfaff's demonstration) $1^{\frac{m}{n}q} = 1$. Hence, when $z = 0$, the product or value of X^Y becomes unity.

In the same volume of Crelle, p. 292, we find two answers to Möbius, one "Von einem Ungenaunten," the other by the author of No. 25. The latter remarks that no one denies that x^x becomes unity when $x = 0$, but if we say that $0^0 = 1$ means nothing more than this, we must also say that $0 \div 0 = 1$ means nothing more than that $x \div x$ is unity when $x = 0$. As to the demostration of Möbius that $[F(x)]^{f(x)}$ becomes unity when $x = 0$, he replies that it rests upon a principle exploded by Hamilton, in the Transactions of the Royal Irish Academy, in 1830. The principle in question is that if X is a function which vanishes with x it may always be assumed in the form Px^m, where P does not vanish with x.

The Article cited from Hamilton is to the effect that it was a commonly received doctrine that a function of x which vanished with x could always be put in the form

$$Ax^{\alpha} + Bx^{\beta} + Cx^{\gamma} \dots$$

in which α, β, γ, &c. are positive and A, B, C, &c. are constant coefficients. (The assumption here spoken of is evidently equivalent to that made by Möbius.) Hamilton instances the function

$$\varepsilon^{-x^{-2}}$$

as incapable of being put in the above form; for, could it be so assumed, (α denoting the least exponent,) we should have

$$x^{-\alpha}\varepsilon^{-x^{-2}} = A + Bx^{\beta-\alpha} + Cx^{\gamma-\alpha} \dots\dots$$

which is impossible, since, when $x = 0$, the first member becomes zero, while the second member takes the finite value A. To show that the first member becomes zero, he considers the value of its reciprocal $x^{\alpha}\varepsilon^{x^{-2}}$ when $x = 0$; or putting $x = 1 \div y$ of $y^{-\alpha}\varepsilon^{y^2}$ when y is infinite. Expanding we have

$$y^{-\alpha}\varepsilon^{y^2} = y^{-\alpha}\left(1 + y^2 + \frac{y^4}{2} + \frac{y^6}{2.3} + \&c.,\right)$$

of which all the terms are positive, and after a limited number all are infinite when y is infinite; therefore

$x^{\alpha}\varepsilon^{x^{-2}}$ is infinit when $x = 0$, and its reciprocal is zero when $x = 0$.

The above demonstration would apply equally well to the function $\varepsilon^{-(1 \div x)}$, and this suggested to the author of No. 25 an example in which X^Y is not unity when X and Y vanish together. Putting this function for X and x for Y he has $(\varepsilon^{-(1\div x)})^x$ of which the value is constant and equal to $1 \div \varepsilon$.

$(\varepsilon^{(1\div x)})^x$ is likewise an example; since it may be regarded as taking either the form 0^0, when x approaches zero from the negative side, or the form ∞^0, when x approaches zero from the positive side.

The other writer who answers Möbius cites the function $1 \div \log x$ which vanishes with x, and yet cannot be assumed in the form Px^m; for, could it be so assumed, we should have

$$\frac{1}{p} = x^m \log x = \log x^{x^m}$$

The demonstration of Möbius applying to the expression

$$x^{x^m}, \text{ we have when } x = 0 \qquad \frac{1}{p} = \log 1 = 0,$$

but by hypothesis p is a finite quantity. Making use of this function he constructs the example $x^{\frac{a+x}{\log x}}$ which takes the form 0^0 when $x = 0$, but it is equivalent to ε^{a+x}, hence when $x = 0$ its value is ε^a.

The condition under which a function may assume the form 0^0 with a value other than unity may be found as follows: Let u^v be the function, u and v vanishing simultaneously. If u vanish with a finite ratio to any finite power of v, say v^n, let $\left.\frac{u}{v^n}\right]_0 = m$; then, since $v^v]_0 = 1$,

$$u^v]_0 = m^v v^{nv}]_0 = m^0 v^0]_0^n = m^0.1^n = 1.$$

But if there is no finite value of n, we fail to establish the equation. Hence the only exceptional cases are those in which

$$\left.\frac{u}{v^n}\right]_0 = 0, \text{ for all finite values of } n.$$

SOLUTIONS OF PROBLEMS IN NUMBER THREE.

SOLUTIONS of problems in No. 3 have been received as follows:

From Marcus Baker, 111, 112, 113, 114 and 118; Dr. H. Eggers, 113; [Dr. Eggers also furnished an elegant solution of 109, which was overlooked in our notice of solutions of problems in No. 2;] E. S. Farrow, 111; Henry Gunder, 111, 112, 113, 114 and 118; Orra Garvin, 111; Flora Henderson, 111; William Hoover, 111, 112, 113, 114, 116 and 118; W. W. Johnson, 114 and 118; Christine Ladd, 111, 112, 114, and 118; Artemas Martin, 111, 113, 114 and 116; Dr. A. B. Nelson, 111, 113, 114 & 116; O. D. Oathout, 111, 114 and 116; Prof. J. Scheffer, 111, 112, 113, 114 and 118; Anna T. Snyder, 111 and 114; S. W. Salmon, 113, 116 and 118; E. B. Seitz, 116.

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111. "Given $\frac{x+y}{x-y} + \frac{x-y}{x+y} = 3\frac{1}{3} \ldots\ldots (1)$, $x^2 + y^2 = 45, \ldots\ldots (2)$ to be solved without the use of an auxiliary unknown."